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THE MATHEMATICS TEACHER

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THE EXTENSION OF CONCEPTS IN MATHEMATICS[1]

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The subject which I wish to outline in the briefest manner is the old problem of introducing into Algebra and Geometry such concepts as, for example, irrational numbers, imaginary numbers, the infinite, non-Euclidean geometry, four-dimensional geometry. This is a matter which must have been, at one time or another, of interest to each of you, and I owe you an apology, or at least an explanation, for talking to you about something with which you are presumably familiar. My reason for choosing this topic is to show how the introduction of all of these concepts is guided, in a general sense, by a few simple and plausible principles. It seemed to me that it might be of some interest to compress into the frame of one short talk the consideration of questions which have presented themselves to you at various times and divers occasions.

In another sense you are entitled to an apology: I must make the confession that the method which I am presenting is possibly not in all respects the best method of introducing the new concepts. This method may be termed the *historic* or *genetic method,* because it traces, rather closely, the historical manner in which the concepts were introduced.

According to this method we assume first only simple concepts used and, according to the gradual development of the science, we introduce new terms and ideas. We follow the development of a branch of mathematics until we reach a point where we need a new concept, and proceed to fit it in a natural manner into the already existing system.

A second method, of much more abstract and more philosophical character, is the famous *axiomatic* or *postulational* method. As Athena emerges in full splendour from the brain of

[1] This paper was read at the meeting of the Illinois High School Conference, at Urbana, Illinois, November 24, 1922.

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Zeus, so now the complete foundation of a science emerges from the brain of the mathematician. The plans for the mathematical structure to be erected are all drawn up with the intention of making later additions unnecessary. A greater degree of unity is attained than is given by the historical method which leaves, from the nature of its procedure, just the least suspicion of "patch-work."

The two methods also differ in that the historical method has at least the appearance of *defining* the new objects while the axiomatic method tends merely to determine the system of axioms and laws which the concepts are to obey, without making any attempt to define the objects beyond the fact that they must obey this system.

We have nowadays many such postulational treatments of various branches of mathematical science.

For pedagogical purposes, in elementary and often in advanced work the historical method—which we shall use—possesses advantages over other methods.

We may imagine that, at a certain stage of the development of mathematics, Arithmetic dealt only with positive and negative integers, including zero, and with positive and negative fractions; and that Geometry dealt only with a finite three-dimensional space of Euclidean character.

At this stage we should have in Arithmetic, or Algebra, no irrational numbers, no complex numbers, no infinity; in Geometry, no infinite elements, no "complex points" or "plane or complex numbers," no non-Euclidean spaces, no four-dimensional or higher-dimensional spaces. An indication of the difficulties which the human mind encountered in introducing new mathematical concepts is, as has been frequently pointed out, contained in the terminology. We mention the terms "surd" (derived from "absurd"), "irrational," "imaginary," "transcendental."

How was the domain of Mathematics gradually enlarged so as to include these new concepts?

There are clearly two questions of outstanding importance in this inquiry:

1. *Why* were the new concepts introduced?

2. *How* was it done? Can we recognize any general rules for such extensions?

We consider separately, first Arithmetic and Algebra, and then Geometry.

Arithmetic and Algebra

Since the introduction of irrational numbers is, in many respects, of more delicate nature than the introduction of complex numbers and of the infinite, we deal first with the complex numbers.[1]

I. *Complex (imaginary) numbers* $a + bi$. The first occasion to introduce complex, or imaginary, numbers arose in connection with the solution of quadratic equations $ax^2 + bx + c = o$, and although, historically, the complex numbers were not introduced at this stage, we shall in this discussion not go beyond these quadratic equations in searching for reasons for introducing them.

We know that if the discriminant $b^2 - 4ac$ is negative, we have no real roots. We are therefore faced by the alternative of either saying that there are *algebraic* equations, and even *quadratic* equations, and even *very simple quadratic* equations, such as $x^2 + 1 = 0$, which have no solution, or, if we wish to maintain the general theorem that every quadratic equation has a root, we must admit these new, complex numbers. We may state this differently: If we are to extract the square root of a negative number, we are confronted with the choice of either saying that "it can't be done" or of introducing the new complex numbers. We may formulate as follows our reason for introducing these numbers—*a reason which we shall meet again and again when extending mathematical concepts*:

[1] For the introduction of the number zero, of negative integers, and positive and negative fractions, which we have assumed already absorbed into our system, it will be sufficient to recall that zero and the negative numbers may be thought of as introduced to extend the scope of subtraction ($a - b$ for $b \geqq a$), and fractions as introduced to extend the scope of division ($a:b$ for a not a multiple of b). The introduction was in each case made in such manner that the fundamental laws for working with positive integers shall still hold after the introduction of the new numbers (for example, the communtative laws of addition and multiplication $a + b = b + a$ and $ab = ba$; the associative laws of addition and multiplication,

$$(a + b) + c = a + (b + c) = a + b + c$$

and $$(a \cdot b) \cdot c = a \cdot (b \cdot c) = a \cdot b \cdot c, \text{ etc.})$$

Reason: In order to extend the scope of an important theorem, or, *in order to extend the field of an important operation.*

It is hard indeed to imagine what would be the status of mathematics, if, every time we had to consider the square root of a negative number, or every time we had to solve a quadratic equation

$$ax^2 + bx + c = 0, \quad b^2 < 4ac,$$

we should find ourselves facing a stone wall and should be forced to drop our investigation right there.

But, once the decision made that complex numbers had to be introduced, how was this to be accomplished? How were these strange aliens to be naturalized and trained to be well behaved citizens in the orderly and law-abiding republic of numbers?

The problem seems to present some difficulties: They are a well-to-do class, possessed of two dimensions, with a possible tendency to lord it over the common reals. Will there be class fights, or can we have a peaceable assimilation of the newcomers? The solution seems plain: They cannot be accepted as citizens unless they conform to the laws of the realm. The constitution must be upheld. This is exactly what has happened with the complex numbers. Merely saying that $\sqrt{-1}$ or i is to be the root of $x^2 + 1 = 0$ is not worth the chalk on the board, until we have stipulated how this new and unknown being is to behave in its relation to other numbers. How will it behave when it meets citizen 5? Will $5 \times i = i \times 5$, that is, will the commutative law of multiplication hold? How will it fit in with the other fundamental operations and laws? Will it still be true that $a \cdot b = 0$ when and only when at least one of the factors vanishes, even when the factors are complex numbers?

The answer to our second question: *How* are the complex numbers introduced? is now nearly obvious: They are defined in such manner that they cause the least possible disturbance. In other words, we aim to define them in such manner that all formal laws for working with real numbers will still hold, after the introduction of the new numbers. Of great importance in this connection is the fact that our real numbers are contained in the totality of complex numbers $a + bi$ as special cases, namely for $b = 0$.

The method of procedure just outlined has been elevated to the dignity of a principle: Hankel's *principle of the permanence of the formal laws.* Therefore,

Method of introducing complex numbers: Preservation of existing laws.

Nearly always, when we introduce new concepts, it is done in consistence with this principle.

I merely mention that there exist some classes of individuals, the socalled hypercomplex numbers, which refuse to conform to some of the basic laws of the realm; at least they have their idiosyncrasies: some insist on never changing sides when walking with one of their fellows: $a \cdot b \neq b \cdot a$, and the commutative law of multiplication does not hold. We are gradually getting accustomed to their fixed ideas.

II. *The Infinite* ∞. This citizen should really be treated as an "undesirable alien." But by hook or crook he has managed to get his naturalization papers, and so we have to make the best of him. But he remains a slippery customer, and, unless watched very closely, will run afoul of our most sacred laws. As a matter of fact, he is so unmanageable that we have had to let him have some laws of his own. Every decent and law abiding number obeys the law that if $a + b = a + c$, then $b = c$. ∞ will rather not be naturalized at all than accept such a law; it is an outrage to his nature and he insists that $\infty + 12 = \infty + 10$, and that $2 \cdot \infty = \infty$. (Also in our terminology this exceptional position of ∞ presents itself for example, in expressions like $\lim\limits_{n=\infty} a_n = \infty$, which is not a *limit* at all, in any strict sense.)

Whatever possessed us to admit such a fellow? We were tempted in various ways. The first temptation we resisted. This arose in the following fashion: If we divide 12 by 4, we have a definite result, 3; similarly, a divided by b, $b \neq 0$, represents a definite number c. However, if we divide any number, for example 5, by 0, $\frac{5}{0}$ does not represent a number. Infinity offered to take the place of this missing number, if admitted to citizenship. After due deliberation the congress of arithmetic voted down this proposition and even passed an amendment to the constitution forbidding by dire punishment division by zero.

This measure was taken because the congress possessed clear proof that behind this apparently innocent division by zero lurk the most sinister anarchistic and nihilistic tendencies; it is indeed a proven fact that if this division by zero is admitted, all laws will fail. But for the enforcement of this special law, everlasting watchfulness is required.

In this particular case, the effort to enlarge the scope of a fundamental operation (division, to include division by 0) has failed disastrously.

Should we not, perhaps, do without ∞ altogether? It might for large parts of mathematics, be possible. But the tempter persisted: By bidding us speculate on the fact that between any two numbers, however close together they stand, we can always squeeze just one more number—just one—; by bidding us reflect on the fact that, however large a number, we can always add one more—just one—; by asking us to ponder over the fact that, however close we be to a point, we can always go still closer without touching it—always room to go just a little closer—; in dozens of such arguments he persuaded us to consider his application, until we finally gave in.— Perhaps one reason for leniency was that we do not consider him full grown and quite responsible for his actions; for, although of no mean size, we always think of him as growing, ever growing, never reaching his full stature · —. Only within the last generation or so have we obtained a glimpse of some of his brethren which have actually attained their full growth; but they form an august clannish crowd which moves and lives in the very refined circles of the theory of assemblages.

What then shall we give as a reason for introducing ∞ ?

> *Reason: It seems unnatural to our mind to assume that there is a largest number, with all that this statement implies.*

And for our method of introducing ∞ ?

> *Method: No number ∞ is introduced. Instead, rules are introduced for dealing with quantities which may grow larger and larger,* (larger than an arbitrarily assigned number.)

If we really want to introduce ∞ as a number, we may, in a certain sense do so. But we must revise all rules of operation and build up a new arithmetic for it. Connected with this ∞ (which has sometimes been called the "static infinite," in distinction to our infinity, the "dynamic infinite," is the question of existence of "infinitesimals," as quantities between 0 and an arbitrarily small positive quantity.

III. *Irrational numbers.* The irrational numbers, as the complex numbers, are law-abiding citizens. However, they are of highly complex mentality. They are citizens of the very highest type, but they expect respectful treatment of their racial characteristics. To the rational citizen of our republic all irrational beings are apt to look alike. But there are really two entirely distinct races of irrational numbers.[1] We can describe these two classes of irrational numbers as follows: An irrational number is every number not of the type a/b, a, b, both integers. One of the classes consists of the numbers which possess the property that there is an algebraic equation with integral coefficients which is satisfied by this number. These numbers are the very important "algebraic numbers," and they include, as special cases, the rational numbers. In particular, every expression formed by performing on rational numbers a finite number of the operations: addition, subtraction, multiplication, division, raising to a power, extracting a root — represents such an algebraic number. (But the vast majority of algebraic numbers are not capable of such representation). For this class of irrational numbers we can give nearly literally the same reasons for their introduction in Algebra and their admission to citizenship, and literally the same method for making them law-abiding citizens, as applied to the complex numbers.

[1]It is a question of taste and purpose whether one prefers to divide the totality of real numbers into *rational* and *irrational* numbers, or into *algebraic* and *transcendental* numbers. The mathematical difficulties are the same. Essential is only that the rational numbers and the algebraic numbers each form a denumerable set, while the irrational numbers and the transcendental numbers each form a non-denumerable set. In other words, it is possible to establish a one-to-one correspondence between the rational numbers and the ordinary integers 1, 2, 3, $\cdots$, and similarly between the algebraic numbers and the integers 1, 2, 3, $\cdots$ but it is not possible to establish a one-to-one correspondence between the positive integers and either the irrational numbers or the transcendental numbers.

Reason: To extend the scope of an important theorem (for example, $x^2 - 2 = 0$ has no root until we introduce the irrational number $\sqrt{2}$); or,
To extend the field of an important operation (the extraction of radicals).

Method: Preservation of existing laws.

The second race of irrational numbers consists of all irrationals which are not the root of any equation with integral coefficients, of however high a degree. They are aristocrats in their own land, including, among their number such august personages as King e and Queen π. These are the famous "transcendental" numbers. They are, at heart, good citizens of the republic of numbers, but you will readily imagine that the process of naturalization has been, in their case, a very tedious and formal process. They retain their aristocratic allures. As far as elementary mathematics are concerned, we should probably never notice the difference if they were not existent. At most, we should be obliged to say that there is no number, and never can be any number, which gives exactly the ratio of the length of the circumference of a circle to the diameter. And after all, this statement would not be so senseless, by any means, as might at first sight appear.

If we are to state in simple words a reason for introducing the transcendental numbers into mathematics, we may perhaps be permitted to fall back on a psychological motive: we are accustomed since many generations to think of the sequence of real numbers, say from 0 . . . 1, as possessing in all respects the same structure as the so-called geometric continuum from 0 . . . 1. However, if we restrict ourselves to algebraic numbers, we feel very soon that something is lacking in the correspondence; there are points — many, many of them — to which no number corresponds. By introducing the transcendental numbers we bridge this gap. In a certain sense, our reason for introducing transcendental numbers is again the endeavor to extend the scope, this time not of a *theorem*, perhaps, but of the *principle* that we can establish a complete correspondence between our arithmetic continuum and the geometric continuum.

It will not have escaped your attention that this reasoning leads us dangerously close to a circulus vitiosus, since one of the main recent tendencies of analysis has been to make analysis and its foundations independent of geometric intuition, and, indeed, to reduce the understanding of the *geometric* continuum to the understanding of the *arithmetic* continuum. However, without geometry to guide our steps it may be considered doubtful whether we should have developed the arithmetic continuum at all.

We state then as a reason for introducing transcendental numbers:

> *Reason:* (psychological) *To extend the intuitive notion of the geometric continuum to the arithmetic continuum.*

The tedious and formal process of naturalization which I mentioned above, consisted exactly in examining whether these high-born aristocrats would be willing to accept in a democratic spirit the laws which govern their humbler brethren: the commutative and associative laws of addition and of multiplication; never to be divided by zero, and to live peacefully in the land. Their examination, carried out by incorruptible and searching analysis, not by the easy going gang of geometric Intuition, has been most minute, but their loyalty has stood every acid test.

> *Method: Preservation of existing laws.*

When we glance over the whole field of numbers and their arithmetic, the integers, the zero, the fractional numbers, the irrational numbers, the complex numbers, what impression do we carry away? The old sturdy stock of the race, represented by the puritanical integers, has gradually dominated and assimilated them all; the laws of operation—the associative, the distributive, the commutative laws, and a host of others—now encircle all the new numbers. They have all had to conform to the age-old laws of the integers.

Kronecker, the German mathematician, said: "Die ganzen Zahlen hat der liebe Gott geschaffen, alles andere its Werk von Menschenhand."

GEOMETRY

Dropping the figurative manner of speech, we continue the subject with the respect it deserves.

When we turn to Geometry, we are confronted with problems of a similar type.

Starting from an assumed Euclidean Geometry for a finite portion of ordinary three-dimensional space, how do we introduce into this geometry the infinite elements, how do we introduce the imaginary elements, how the four- and higher-dimensional spaces?

IV. *Infinite Elements in Geometry.* The desire to introduce infinite elements in geometry would seem to be nearly unavoidable on account of our psychological conception of space. It seems intolerable to us to assume that we cannot, in imagination if not in body, go in any given direction as far as we like, intolerable that there should be a greatest distance which cannot be exceeded. [It is only at a stage of great mathematical refinement that the possibility of a space which is finite in all directions is conceived—the Riemann space—and even this space is infinite if lengths are measured, not in our Euclidean manner, but by conventions which are natural for the Riemann Geometry]. To return to our ordinary plane: We are less interested in the psychological reasons for introducing infinite elements in Geometry than in the technical mathematical reasons. If we are dealing with a straight line, we shall introduce as infinite elements all points on the line lying farther away from any given point than an arbitrarily large distance. The definition follows closely the corresponding definition in Algebra.

Similarly, in a plane, the infinite elements are formed by the points lying farther away from any given point of the plane than an arbitrarily assigned distance. And similarly for space.

The independence of character of the Infinite which we learned to fear and respect in Algebra reasserts itself in Geometry, this time in the sense that it insists on being treated differently on different occasions.

(*a*) *The Infinite in the Euclidean plane* (*and in the Projective plane.*)

We select as paradigma the plane; for space everything mutatis mutandis.

If we consider two straight lines in a Euclidean plane, they always have exactly one point of intersection, unless they be parallel. It so happens that by introducing in an appropriate manner infinite elements in the plane, we can remove this exceptional case. I need not insist on this point. We all know that this removal of the exceptional case is effected by the fiction that two parallel lines meet "at infinity." Using in repetition the fact that two lines are to have always exactly one point in common and the fact that through two points exactly one line is determined we find that our elements at infinity must be constituted as follows:

(a) On every straight line there is exactly one point at infinity (not two, one for $+\infty$ and one for $-\infty$; in distinction to the infinite of arithmetic). This point at infinity lies at infinite distance from every finite point.

(b) Any set of parallel rays in a plane have altogether just one point at ∞ (to every direction belongs one point ∞) — To two different directions belong two different points. — Therefore there are, roughly speaking, as many points at infinity as there are different directions in the plane, that is, there are ∞^1 points at infinity. These ∞^1 points all lie on a straight line and fill it out: the line at infinity, which nowhere penetrates into the finite part of the plane. Every part of this line is infinitely distant from every finite point of the plane.

(c) Similarly, in space we have a whole infinite plane, of which every point is infinitely distant from every finite point of space. Every other plane cuts this plane in a line at infinity; etc., etc.

For introducing ∞ in Euclidean (or in projective) space we may then say:

> *Reason:* (a) (psychological reason) *It seems unnatural to our mind to assume that there is a limit to our space.*
>
> (b) (mathematical reason) *To extend the scope of an important theorem* (two distinct lines in a plane always have exactly one point in common).

Concerning the method of introducing the infinite elements we may say:

> *Method: Preservation of existing laws;* that is, so that the elements at ∞ will fit in with the finite elements without conflicting with existing laws.

β) The Infinite in the plane of complex numbers of analytic functions; in the plane of the Geometry of Inversion.

Without enquiring into the reasons for, and the method of, introducing a plane of complex numbers, we shall only state how in this plane, once it is accepted, the infinite elements are introduced. It is only necessary to point out that in this Geometry the straight lines and their properties count for little; it is the algebraic transformation of the plane into itself by menas of the transformation $z' = {}^1/_z$, and by its big sister $z' = \dfrac{\alpha z + \beta}{\delta z + \psi}$ which dominates this field. By the transformation $z' = {}^1/_z$ a point outside the unit-circle is transformed into a point inside the circle, and vice versa. With one exception, however, which is here just as important as in the Euclidean plane the breakdown, in the case of parallel lines, of the property that two lines have exactly one point in common. This exceptional case arises from the fact that to the origin, the centre of the circle, corresponds no point outside the circle. But the fact that to the interior of any very small circle with the origin as centre corresponds the whole exterior of a very large circle about the origin has lead us to the following assumption: In the complex plane (and in the plane of the Geometry of Inversion) the whole exterior of a circle about the origin (or about any finite point of the plane) whose radius can be made greater than any assigned number, corresponds to the one point at the origin (or to the center of the circle), and is therefore called "the point at ∞" of the complex plane. In other words in the complex plane, we have only one point at infinity, and we "approach" the same point at ∞ in whatever direction we start out;—"All roads lead to Rome." For For those among you who are familiar with stereographic projection, this assumption concerning the character of the point at ∞ will appear more reasonable.

Reason for introducing infinite elements in the complex plane: *To extend the scope of an important theorem* (Under inversion—or under the general linear transformation—to every point of the plane corresponds exactly one point of the plane.)

Method: Preservation of existing laws.

We notice the following point: although we have introduced two, mutually exclusive, interpretations of the infinite (in the Euclidean plane and in the complex plane), we must never take this to mean that one interpretation is true and the other false. We shall meet a similar situation, in more pronounced form, in Non-Euclidean Geometry.

V. *Non-Euclidean Geometry.*[1] It would require several hours to justify in detail the introduction, alongside of our Euclidean Geometry, of the so-called Non-Euclidean geometries: the Bolyai-Lobatschefskij Geometry and the Riemann Geometry. A few generations of comparative familiarity with these amazing conceptions of the mathematical mind have probably sufficed to dull our astonishment at these wondrous new spaces. "Familiarity breeds contempt." To emphasize in only one point the far-reaching importance of their introduction: It is, I think, extremely probable that our mind would not have been willing seriously to consider Einstein's world of relativity if we had not undergone the mental training which the introduction of Non-Euclidean Geometry exposed us to. Even if we did not completely absorb the principles of Non-Euclidean Geometry—there are too many race-old prejudices to be overcome—yet it put us into a more or less receptive frame of mind.

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[1] A superficial, but annoying, source of confusion lies in the fact that the term *Non-Euclidean* is sometimes used in a wider sense, so that it includes more than the geometries considered under V. For example, the four- and higher-dimensional geometries are often counted into Non-Euclidean geometry. Unfortunately, these writers have a strong formal argument in their favor: every geometry which differs from the ordinary Euclidean three dimensional geometry is really non-Euclidean. But the term has come, in the minds of the majority of mathematicians, to mean that a geometry is Euclidean or Non-Euclidean according to whether the curvature is zero or different from zero, regardless of the number of dimensions. Most of the elementary illustrations of Non-Euclidean geometries are indeed carried through for two-dimensional spaces, not for three-dimensional ones.

A four- or higher-dimensional geometry is Euclidean or Non-Euclidean, according to whether its curvature is zero or different from zero.

We ask again our old questions:

Why were the Non-Euclidean geometries introduced?

How were they introduced?

The "why" of Non-Euclidean geometry is of extreme importance for the clarification of our ideas on what constitutes mathematics. To mention one phase: the powerful and important method of investigating the foundations of mathematics which we have already referred to as the axiomatic or postulational method had here its origin and found here the source of its strength. I remind you of the famous "Parallel Axiom" according to which there is, in a plane, exactly one line parallel to a given line and passing through a given point not in the line, or, in other words, there is always exactly one such line which does not intersect the given line. (Of course we are not admitting here points of intersection at infinity which are, after all, fictitious).

We cannot review here the reasons why this axiom has always enjoyed a special position in man's mind, as compared with the other Euclidean axioms. Through the ages, from Euclid down, we can trace a haunting feeling that this axiom stands apart from its kind and we witness a long series of efforts either to replace it by some simpler axiom or to prove it a consequence of the other axioms. Finally, in the early nineteenth century, the Hungarian Bolyai and the Russian Lobatschefskij proved that if we leave all other axioms unchanged and replace the parallel axiom by a certain other one which contradicts it, viz., the axiom that it shall be possible to draw through a point outside a line (in the plane) more than one line not intersecting it (and therefore an infinite number of such lines) we can then again build up a geometry which contains no logical contradiction in itself. *This is sufficient for us to admit it as a valid geometry, in every mathematical sense as real and as valid as the Euclidean Geometry.* The question, why we cling so tenaciously to Euclidean Geometry as the "really, truly, honest to goodness" Geometry is an interesting and important question, but one which does not belong to the domain of mathematics.

Later the German mathematician Riemann showed that we can build up another geometry, likewise free of internal logi-

cal contradiction, if we replace the parallel-axiom by the assumption that *no* line can be drawn in the plane through the point which will not intersect the given line. This assumption leads to the Riemann Geometry, referred to above as "finite."

We have here a very curious state of affairs: Three geometries (or, rather, two types of geometries, the Bolyai-Lobatschefskij type and the Riemann type, and an intermediate geometry, the Euclidean Geometry), each of which is mathematically sound, each from the standpoint of every mathematical and logical test, true, and yet contradicting one another.

By rising to a certain height of abstraction we satisfy ourselves without trouble that the situation contains nothing to disturb our peace of mind. We can, for example, overcome our difficulty in this manner: If we select any particular one of the Bolyai-Lobatschefskij geometries, it is characterized by the existence of a certain so-called "negative curvature," which is measured by a certain corresponding negative number. If we choose any other negative number, then to it corresponds another Bolyai-Lobatschefskij geometry, of different curvature from the first. Similarly, if we select any particular Riemann geometry, it is characterized by the existence of a certain "positive curvature" which is measured by a certain positive number. If we choose another positive number, then to it corresponds another Riemann geometry of different curvature. It is for this reason that we talked before of "types of B.-L.-geometries" and "types of R.-geometries." Sandwiched in between the B.-L. geometries and the R.-geometries, as the zero is sandwiched in between the positive and the negative numbers, is our Euclidean Geometry, whose curvature is measured by the number 0.

We can therefore, in our inner mind, imagine these spaces all strung out in a linear arrangement—or perhaps we cannot—just as we may imagine the real numbers all strung out, and, exactly as we comprise the totality of real numbers into the class of real numbers, just so we may consider a kind of absolute geometry which includes all of the Non-Euclidean geometries (B.-L. and R.) *and* the Euclidean Geometry, as special cases. In an abstract mathematical sense, this is the easiest thing in the world to do. We are merely introducing the number measur-

ing the curvature as a new "parameter" or "dimension." Now the various geometries do not exclude each other in any other sense than the sense in which the real number 2 excludes the real number 3. We have said enough to justify the following answers to our questions:

Reason for introducing Non-Euclidean geometries: *Logical difficulties encountered in the examination of Euclid's axioms.*

Method: (a) Establishment of a whole range of non-Euclidean geometries, including among them the Euclidean Geometry.

We note that our Euclidean Geometry is obtained from this absolute geometry, by choosing for the curvature the measure 0 in logically the same sense in which we obtain from the totality of complex numbers $a + bi$ the real numbers by choosing $b = 0$. We should thus be justified, in close analogy with the case of complex numbers and other cases, to add, under

Methods: (b) Preservation of existing laws.

VI. *Four-dimensional Geometry*[1]. The last subject to be considered is four-dimensional geometry. In this geometry, we have again to distinguish well between the mathematical existence of such a space (which involves nothing but lack of inner contradiction in the system), and the possibility of something corresponding to this four-dimensional geometry in the same sense in which our every-day three-dimensional space corresponds to three-dimensional geometry. Be it merely mentioned that this question of the "reality" of four-dimensional space is to the mathematician, for various reasons, of less interest than the corresponding question in the B.- L.- and R.- geometries, and we know already that even there the question is really not a mathematical problem. The mathematician rests satisfied, we repeat, if he can establish a "four-dimensional geometry," even if it have no realization outside of his formulas and constructions, provided only that these formulas and constructions can never lead to a logical contradiction within the system. Indeed

[1]Euclidean. See footnote to V.

he finds not much more difficulty in constructing a geometry of 5 or of n dimensions, and is somewhat at ease in a "function space of infinitely many dimensions."

It is, for our purposes, satisfactory to say that the mathematician invents, for purposes of convenient expression and as a valuable aid in the discovery of new theorems, a new geometry which corresponds to our 3-dimensional geometry in the same manner in which our 3-dimensional geometry corresponds to the geometry of the plane.[1] Methods of generalizing from our 3-dimensional geometry to a four-dimensional geometry suggest themselves readily, and since it can be shown that the new system, irrespective of any spatial interpretation, is consistent in itself, it is a matter of small wonder that a geometrical nomenclature and a geometric interpretation can be invented which are also not self-contradictory. And this, we know, is all that is required in order to be able to speak of 4-dimensional space in a mathematical, but not in a psychological or metaphysical, sense. (This is one extreme way of looking at the problem. There are other directions of attack, but probably no responsible mathematician now-a-days puts forth any claims for the reality of four-dimensional space in the sense in which we are accustomed to think of three-dimensional space. Of course, anybody who has so thoroughly absorbed Einstein's theory of relativity that he feels quite at home in Einstein's time-space world has a right to argue that for him the question of a real 3-dimensional world has no meaning at all.)

You are familiar with the interpretation of formulas in analytic geometry of one, two, three dimensions. Let us then go one step farther, into the fourth dimension. We find at once:

(x_1) = coordinate of a point in a line (in 1-dimensional space);

$(x_1\ x_2)$ = coordinates of a point in a plane (in 2-dimensional space);

(x_1, x_2, x_3) = coordinates of a point in a space (in 3-dimensional space);

(x_1, x_2, x_3, x_4) = coordinates of a point in a space of 4 dimensions.

[1] The importance of a convenient symbolism must not be underrated. In most of modern mathematics, progress depends vitally on the existence of an adequate symbolism.

Or:

$\sqrt{(x_1 - y_1)^2} = |x_1 - y_1| =$ distance between x_1, y_1 in one dimensional space;

$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} =$ distance between $(x_1, x_2), (y_1, y_2)$ in two dimensional space;

$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} =$ distance between $(x_1, x_2, x_3), (y_1, y_2, y_3)$ in three dimensional space;

$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2 + (x_4 - y_4)^2} =$ distance between $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)$ in four dimensional space.

Similarly:

$(x - a)^2 + (y - b)^2 = r^2$, equation of a circle in two-dimensional space;

$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ equation of a sphere in three-dimensional space;

$(x - a)^2 + (y - b)^2 + (z - c)^2 + (w - d)^2 = r^2$, equation of a hypersphere in four-dimensional space,

or, to mention one more set of formulae,

$\frac{x}{a} = 1$, equation of a point in one-dimensional space;

$\frac{x}{a} + \frac{y}{b} = 1$, equation of a line in two-dimensional space;

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, equation of a plane in three dimensional space;

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{w}{d} = 1$, equation of a hyperplane in four-dimensional space.

It is as simple as all that!

Just as, in three dimensions, two planes intersect in a straight line (barring exceptional cases), just so, in four-dimensional space, two hyperplanes intersect in an ordinary two-dimensional plane (barring exceptional cases).

We may already guess, from these illustrations, that we obtain our three-dimensional space from the four-dimensional space by

letting one dimension vanish. The importance of this we readily appreciate when we remember other cases where an analogous situation arose.

We may also approach the fourth dimension from a more geometrical angle: We can reconstruct a model in three-dimensional space from its two-dimensional picture, that is, from its projection from three-dimensional space onto two-dimensional space; in the same way we may ask, starting out from a given model in three-dimensional space, from what kind of four-dimensional object it may have been generated by projection from a point in four-dimensional space onto a three-dimensional space; or, in other words, of what four-dimensional solid it may be considered a three-dimensional shadow! We note with particular care that this in no way implies the *existence* of four-dimensional space. In fact, we may formulate the problem as follows: *If* there exists a four-dimensional space, then what will the solid in it be like which gives as its three-dimensional shadow our given solid?

That we are able to make definite statements concerning "mathematical" objects without knowing whether they "exist" at all or not, is to the mathematician a familiar fact. For example, in the theory of numbers we speak of a "perfect number" as a number of which the sum of all proper divisors is equal to the number itself: 6 is a perfect number, because $1 + 2 + 3 = 6$; 28 is a perfect number because $1 + 2 + 4 + 7 + 14 = 28$. These two perfect numbers are even numbers. A few more *even* perfect numbers are known. Concerning *odd* perfect numbers, we know quite a few properties. To quote only one simple theorem, every odd perfect number is of the form $4k + 1$. *But we do not know whether any odd perfect numbers exist!*

To give an example of a three-dimensional shadow of a four-dimensional solid, we state that if we hang into a wire model of a cube a second small cube, so that the cubes have the same centre and their corresponding pairs of edges parallel, and join through eight pieces of wire corresponding pairs of vertices, then this model (consisting of 32 pieces of wire) is the "three-dimensional shadow" of the regular four-dimensional "8-cell" ("bounded" by 8 three-dimensional cubes—the large

cube, the small cube, and six more "cubes," which are distorted in our model into truncated pyramids).

We have perhaps said enough to be able to answer our ever recurring questions:

> *Reason* for introducing four-dimensional space:
>
> (*a*) *Convenience of expression;* with the closely related economy in thinking.
>
> (*b*) *To extend the scope of geometric interpretation of formulas.*
>
> *Method*: *Preservation of existing laws.* (Our three-dimensional space fits into the enlarged scheme).

Conclusion Glancing over our table we find evidence of a principle which seems to govern the introduction of important new concepts into mathematics. We find again and again that new concepts are introduced for the reason that we wish to extend the scope of an important theorem or operation, frequently in order to remove exceptional cases which made it impossible to state fundamental theorems in the full generality they deserve.

The *method* of introducing the new concepts shows even greater uniformity than do the *reasons* back of introducing them at all. Once it is decided that the new concept is to be incorporated into the system of mathematics, the utmost care is taken that nothing which is of value in the already developed theory shall be disturbed.

The builders of mathematics are ever ready to incorporate new ideas, but not at the price of sacrificing a jot of the solid foundations on which their proud and many towered castle is being erected.

RÈSUMÈ

ARITHMETIC AND ALGEBRA.

	Complex Numbers	*The Infinite, ∞*	*Irrational Numbers* — *Algebraic*	*Irrational Numbers* — *Transcendental*
REASON:	To extend the scope of an important theorem; to extend the field of an important operation.	(Psychological) It is unnatural to our mind to assume that there is a largest number.	To extend the scope of an important theorem; to extend the field of an important operation.	(Psychological) To extend the intuitive notion of the geometric continuum to the arithmetical continuum.
METHOD:	Preservation of existing laws.	No number ∞ is introduced. Instead, rules are introduced for dealing with quantities which may grow larger than an arbitrarily assigned number.	Preservation of existing laws.	

GEOMETRY.

	The Infinite — *Euclidean and Projective plane*	*The Infinite* — *Plane of Complex Numbers*	*Non-Euclidean Geometry*	*Four Dimensional Geometry*
REASON:	(a) (Psychological) It is unnatural to our mind to assume that there is a limit to our space. (b) (Mathematical) To extend the scope of an important theorem.	To extend the scope of an important theorem.	Logical difficulties encountered in the examination of Euclid's Parallel Axiom.	(a) Convenience of expression; economy in thinking. (b) To extend the scope of geometric interpretation of formulae.
METHOD:	Preservation of existing laws.		(a) Establishment of a whole range of Non-Euclidean geometries, including among them the Euclidean Geometry. (b) Preservation of existing laws.	Preservation of existing laws.

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AN EXPERIMENT IN CLASSIFICATION OF PUPILS IN ALGEBRA[1]

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It is the purpose of this paper to describe an experiment which is being conducted at the New Trier Township High School, in segregating Freshmen into slow-moving, normal and accelerated classes in Algebra, an experiment which is undoubtedly being duplicated in other schools. We are describing our experiment with the hope of bringing about a discussion as to the advisability of such classes, best method of conducting them, subject-matter to be taught, and results that may be expected.

At New Trier, for the past year, we have been conducting two slow-moving groups and one accelerated group in addition to seven normal groups. We have been making this experiment, because we have felt very keenly that the old method of arranging classes without regard to mental ability was extremely inefficient, since it resulted in wasted time for both the accelerated and slow-moving groups. In the case of the slow-moving group, a group usually entirely among the failures, we feel that the loss was not only a loss in actual time—but the much greater loss of mental training and discipline—since the habit of failing, once acquired, is such an extremely hard one to combat and one which carries with it dangerous mental habits and attitudes toward work. We therefore decided that something really accomplished, no matter how small, was a distinct advantage over our old methods of caring for, or failing to care for, the mentally slow.

The first task which presents itself in connection with such a course, is the method of selection. Here we must exercise the most careful judgment and make schedule arrangements so elastic, that shifting back and forth can be easily arranged. For, in spite of all we are learning concerning mental tests, we are forced to admit that they are not absolutely reliable aids in pigeon-holing boys and girls.

[1] Paper read before Mathematical Section of High School Teachers' Conference at University of Chicago, May 12, 1922.

The method employed for selection was as follows: Mental tests were given in the spring in the eighth grades of the schools in our High School district. The boys and girls were then ranked, not only according to their I. Q. from mental tests but also according to their previous school records. From those at the head of the list, an accelerated group was formed and from the end of the list, the two slow-moving groups. Of course, new pupils entering our school-district for the first time, were not properly located by such a method—moreover, we soon discovered that among our normals, there were slows, and among our supposedly slows there were normals. Consequently, it was decided to carry all of our classes along at a normal rate for the first month, and then after a close study of each case make necessary shifts. (I might add that most of the shifting was for those who were entering our school district for the first time. Except for two or three unusual cases, the mental tests were a satisfactory method of grouping.) We found this an extremely satisfactory method of selection, until along toward the last of the first semester, when some of our plodders, having been allowed to progress slowly, showed signs of brilliance and were transferred to the normal group for the second semester. I have two such notable examples, who are doing 90-95% work, now, in normal sections. We attribute their excellent work this semester to the fact that they were allowed to go slowly at first.

In order to make this shifting from one class to another as simple as possible, our schedules were arranged so that a slow-moving and normal group were in session at the same time. This arrangement, of course, permitted shifts to be made without disturbing other classes.

The personnel of the slow-moving classes easily divides itself up into three groups: (1) those who are slow in grasping an idea—but when they have grasped it, retain it. (2) those who are slow in grasping an idea and do not easily retain it and (3) those who are mentally lazy. This latter group, we have kept as small as possible by transferring them, if at all possible, into the normal group and attempting to awaken them from their mental lethargy. They have been in the minority in our slow-moving classes.

In order to give the greatest assistance to these groups of individuals, the kind of instruction and the material used has had to be carefully chosen. We have found that the time for explanation has had to be increased. In fact, an explanation in the slow-moving group must frequently take four or five times as long as in a normal group—and, moreover, be frequently repeated. Daily and weekly drills in the operations supposedly mastered were necessary to assure mastery of the material.

The material used, was of course, much more simple, but the ground covered, as far as subject matter is concerned, was practically the same for both normal and retarded groups. The greatest limitation, as far as subject matter was concerned, was of necessity in the formation of equations. Otherwise the subject matter covered was the same.

Out of our two slow-moving classes, numbering 44, the very poorest of the Freshman class, we had 7 failures. All of this latter number are repeating Algebra this semester and with the exception of one, who seems to be hopeless from a mathematical stand-point are doing passing work in normal groups.

As a result of the year's experiment, we exhibit the following statistics in comparison with the previous year.

During this past fall there were enrolled in first semester Algebra, 263 of which 32 failed and 5 withdrew. Only one of the withdrawals was due to discouragement because of failure—the rest, being due to removal from school-district. The percentage of failures was 12.1% and of withdrawals 1.9%.

During the corresponding period of the previous year, there were enrolled 273 of which 36 failed and 16 withdrew, eight withdrawing because of becoming discouraged thru failure. The percentage of failures was 13.2% and of withdrawals 5.8%.

Our greatest improvement is in the percentage of withdrawals an item which makes us feel the worth-whileness of our experiment. The reduction in the percentage of failures is not as convincing as we should like to have it—but in almost every case, our failures this year were due to laziness rather than lack of ability in our normal classes. We feel that these failures were not due to our failure in presenting subject matter beyond the ability of boys and girls to grasp, but rather to the laziness of the student.

The objection which has always been raised to segregation of the slow and normal, was that the element of competition was removed—and that in a slow group, the slow would become slower and the class most uninteresting. The incentive for better work was removed by removing those of superior ability.

I contend that the greatest incentive for good work, is the knowledge that we are capable of doing the task assigned and the presence of people of superior ability is more frequently discouraging than encouraging.

In a slow-moving group, the subject matter is covered so thoroughly and explanations made so frequently that the members of the group know they can do the task assigned and consequently do it. The same amount of explanation, if given in a normal group, while absolutely vital to the slow, would be fatal to the upper group. Furthermore, the poorer students dislike to ask questions in a mixed group, for fear of displaying their ignorance, while in a group of those of equal mental ability, that hesitancy is lost and their understanding of the subject matter is much greater.

The element of competition, is moreover, not lost, since even in such a group there is a great variety of ability and always some who are leaders. Competition between those who are nearly equal in ability—no matter to what group they belong, is truly a great incentive, but between those who are not equal in ability, it is discouraging to the poor and of no value to the strong.

The only hardship experienced in segregation is that the leaders in the slow-moving group do not have those of superior ability with which to contend. This objection, however, can be off-set by constantly presenting to them the incentive of being transferred to a normal group if their work is of high enough caliber.

In the normal group, the absence of "dead or dying timber" improves the morale of the whole group. Where everyone is working and thoroughly understanding what they are doing the lagards are few and the quality of all work improved.

It is our aim to prepare our boys and girls to meet the competition which life forces upon them. I believe that we give

them a better preparation in a class where the instruction can be given so that they really master the material—than in a class where the instruction is over their heads and their knowledge, if they manage to get any—is only sufficient to "pull them through."

In the teaching of such a class, a teacher's ingenuity is frequently taxed by trying to find ways and means of presenting the material in an understandable manner. Her patience is not as sorely taxed, as when she is teaching a mixed group—because she knows of course what kind of a group she is handling and "forewarned seems to be forearmed." Furthermore, the evident appreciation upon the part of the boys and girls—because they are really learning something, is a very great reward for the hard work on the part of the teacher.

In conclusion, I wish to state that we have found two conditions absolutely essential to satisfactory work with a slow-moving class. (1) Small classes—at least less than 25; (2) an elastic schedule permitting shifting when necessary.

I might add also that Miss Maloney of our Mathematics Department and I, who have each had a slow-moving and normal class in Algebra this fall, agree that the segregation of Freshmen is a much more satisfactory course than our old method of hit or miss classification.

PERMUTATIONS IN THE 16TH CENTURY CABALA

MORRIS TURETSKY

INTRODUCTORY NOTE. In the November, 1922 number of the Mathematics Teacher attention was called to some interesting material relating to permutations and combinations which had been discovered in certain Hebrew works by Messrs. Ginsburg and Turetsky. The latter came across his material in a certain work by Moses Cordovero, a learned rabbi of the first part of the 16th century—the Pardes Rimmonim (Orchard of Pomegranates), first printed in the Hebrew language at Salonika in the year 1552. It sets forth the simple and interesting way in which Cordovero attacked the subject of permutations and suggests a method which could be used with profit at the present time. The translation is interesting because of the quaint style, and the notes will be found valuable not merely for their explanations but for the mathematical principles set forth.

DAVID EUGENE SMITH.

That there is a close relation between the Cabala and mathematics is clearly evident. Cabala—the mystic science concerning God, the universe, and divine secrets—included the belief in the creative powers of letters and numbers, magic powers having been attributed to the letters of the Hebrew alphabet. The different letters were considered as powers such as, if suitably combined, could subjugate the forces of nature. This belief naturally led those who indulged in mystic discussions to attempt to ascertain the number of ways in which different groups of letters could be arranged—resulting in a study of permutations and combinations. Thus we find that the *Sefer Yetzirah* (*Book of Creation*), the authorship of which is absurdly attributed by legend to the patriarch Abraham, and which is probably a product of the first few centuries of the Christian Era, contains the results for the permutations of 2, 3, 4, 5, 6 and 7, elements taken all at a time.[1] One of the many books in the rich cabalistic lore which followed the *Sefer Yetzirah*, and was largely based on the tenets laid down by the latter, is the *Pardes Rimmonim*[2] (Orchard of Pomegrantes) of Rabbi Moses Cordovero.[3] The work consists of thirty-two chapters, one of which (Chapter 30)

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[1] See Sefer Yetzirah, Chapter IV, p. 12.

[2] First edition published in Salonika in 1552. Other editions: Venice (1586), Krakow (1592), and Koritz, Russia (1786).

[3] Moses Cordovero was one of the most prominent cabalists of the 16th Century. He was born in Safed, Palestine, in 1522, probably of a Cordoban family, as his name suggests. He was a rabbi in Safed and died there on June 25, 1570.

is devoted to a discussion of permutations and combinations. This article will simply concern itself with his treatment of permutations.

Cordovero begins the chapter "On Permutations" by quoting the passage from the *Sefer Yetzirah,* wherein a statement is made regarding permutations, several special results being given without demonstration. Using the figurative language of the *Sefer Yetzirah*, Cordovero introduces the quotation by a remark that, "A house cannot be built with one stone," implying by this statement that one letter cannot be permuted.[1] He then proceeds to prove and generalize the results given in the *Sefer Yetzirah.*

"Two stones[2] build two houses,[3] for in permuting two letters, two words will be formed each being considered one house; because two letters can be combined in two ways, namely, *ab* and *ba*,[4] for the word, when rolling in a circle,[5] will reverse itself, the beginning becoming the end, and the end replacing the beginning. Three stones build six houses, as follows: *abc* and *acb* (two permutations beginning with *a*), *bac* and *bca* (two beginning with *b*), and *cab* and *cba* (two beginning with *c*). Hence together there are six houses. Four stones build 24 houses, which is illustrated in the following way: Considering the letters *abcd*, there will be six constructions beginning with *a*, namely, *abcd, abdc, acbd, acdb, adbc,* and *adcb;* six beginning with *b*, namely, *bacd, badc, bcad, bcda, bdac, bdca;* six beginning with *c*, namely, *cabd, cadb, cbad, cbda, cdab, cdba;* and finally six beginning with *d*, namely, *dabc, dacb, dbac, dbca, dcab, dcba.* Together then there are 24. If we should examine these permutations we would find them ordered in such a way that each of the above mentioned groups contains the permutations of the other three stones . . . We may infer then that in dealing with a group

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1 While he considers one letter to have no permutations, he considers the number of permutations of a group of the same five letters to be one.

2 Elements.

3 Permutations.

4 Cordovero always uses the letters of the Hebrew alphabet. The translation has been slightly amplified to express more fully the meaning.

5 Pacioli (Suma, 1494) and Tartaglia (1556) consider a somewhat similar case of a number of persons sitting around a table. If a circle is used the letters may be arranged thus: $\overset{a}{\underset{b}{\mathrm{O}}}$ or thus: $\overset{b}{\underset{a}{\mathrm{O}}}$.

of four, five, or six letters, it is useful to first consider the last two stones as forming a group of their own and to permute them as such."

Cordovero then proceeds to illustrate the value of this suggestion in permuting the letters *abcd*.[1] The last two letters yield two forms and the last three letters (*b*, *c*, *d*) give rise to six different permutations. He then places *a* at the head of each of these permutations and has six beginning with *a*. He then states that in a similar way *b* can be placed at the head of the six permutations of *acd*, and *c* at the head of the six permutations of *abd*, and *d* at the head of the six permutations of *abc*. Together we have the 24 permutations of the four letters *a*, *b*, *c*, *d*. Summing up this discussion, Cordovero gives the following rule:

"By looking into the above permutations we discover that, while two stones yield but two, when another stone is added, we will have three times the previous two, since the third stone is cyclically[2] permuted with the other two stones and as a result we shall have two times three, or six. Proceeding thus, four will give four times six, or 24; five will give five times 24, or 120; six will give six times 120, or 720; seven will give seven times 720, or 5,040. From this point on we might similarly compute until we reach the point where there exists no more number."[3]

In the rule just derived all the things have been regarded as "unlike."[4] In the follow section, Cordovero proceeds to

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[1] Cordovero does not use the Hebrew letters for **a, b, c, d;** in this instances he uses the four letters of the name **Shlomoh** (Solomon).

[2] See note 2 on page 33. Most of the modern writers of algebra textbooks who consider permutations look at circular permutations from a different angle. For example, they obtain $4! \div 4$ for the circular permutations of four objects taken four at a time; since in the cyclic arrangement $d \overset{a}{\underset{c}{\;}} b$, abcd bcda, cdab, and dabc are considered one arrangement instead of four distinct permutations as Cordovero considers them.

[3] Altho Bhaskara (c. 1150) gives the rule for the permutation of **n** things taken **r** at a time (**Lilavati**, Colebrooke translation, pp. 49, 123) the first known evidence of interest in the subject, found in print, is in Pacioli's **Suma** (1494), where the author finds the number of permutations of people sitting around a table. (Fol. 43, v.) He gives the results for **n** = 1, 2, ..., 11. Tartaglia, in 1556, also discusses the case of people sitting around a table and gives the results for **n** = 1, 2, ..., 12. The first work of any extent where this subject was discussed is Bernoulli"s **Ars Conjectandi** (1703, Pars Secunda, Caput I).

[4] The terms "like" and "unlike" are here used to denote letters that are respectively visibly alike and are easily distinguishable from one another.

consider and devise a rule for the permutation of the letters of words in which one or more stes of letters are "like."[1]

"If we should find a repeated letter, then one half of the number corresponding to the number of elements should be subtracted. For this reason the tetragramaton will not give more than 12 constructions,[2] or one half the number otherwise resulting—since four stones usually build 24 houses, while these four stones build but 12. . . . A word of three letters would by the law give rise to six forms; if, however, there are two like letters in the group there will be but three permutations. If, in a group of four, a letter be repeated three times, the first duplication will destroy half of the permutations, as we have explained; and the second duplication (that is, the third like letter) will destroy two-thirds of the remainder, four forms remaining out of 12.[3]

"Thus *abbb*, which should give 24 permutations by the original law, will (because of the three like letters) yield but four. In a group of five letters whose permutations would ordinarily be 120, if there are two like letters as in *abcdd*, there will remain 60[4]; if there are three like letters, as in *abddd*, two-thirds of the remaining forms will disappear and but 20, or one-third, will remain as a result of the effect of the third letter[5]; if there is a fourth like letter in the group, then only one-fourth of the latter result will remain. For example, in the case of *adddd*, nobody can make it yield more than five permutations, or one-fourth of

[1] This case in permutation is taken up by Bhaskara (c. 1150) (**Lilavati**, Colebrooke translation, p. 123), who gives a rule for finding the number of permutations of **n** things taken **r** at a time with repetitions. The **Ars Conjectandi** (1713) of Beroulli treats the matter fully. (Pars Secunda, Caput I).

[2]That is, $n = \dfrac{P_4}{P_2} = \dfrac{24}{1\cdot 2} = 12.$

[3]That is, $n = \dfrac{P_4}{P_3} = \dfrac{24}{1\cdot 2\cdot 3} = 4.$

[4]That is, $n = \dfrac{P_5}{P_2} = \dfrac{1\cdot 2\cdot 3\cdot 4\cdot 5}{1\cdot 2} = 60.$

[5]That is, $n = \dfrac{P_5}{P_3} = \dfrac{1\cdot 2\cdot 3\cdot 4\cdot 5}{1\cdot 2\cdot 3} = 20.$

the 20.[1] If there should be a fifth like letter in the group, then there will remain only one-fifth of the previous number, hence we would not have more than one permutation; which is one-fifth of five. Thus *ddddd* can appear in one form only."[2]

Summing up Cordovero establishes the following rule:

"If there are only two like letters, one-half the permutations will remain; in the case of three like letters, only one-third will remain of the number obtained in case of two letters; if there are four like letters, one-fourth of the previous amount will result, and similarly with five like letters, six like letters, seven like letters, and so on.

"This rule, however, is applicable only if all the repeated letters are similar to the first two like letters. If, however, the repeated letters are as in the word *aabb,* a different rule must be applied. The first duplication will destroy half the permutations as previously stated, . . . 12 remaining; and the duplication of the second letter will further reduce the number to six, or half of 12. . . . So, of the letters *aabb,* it is possible to construct only six permutations or half of the 12 perviously derived."[3]

Cordovero then takes a group of five letters in which there are two distinct pairs of like letters, as *abbcc,* and shows that the actual number of permutations is 120 divided by two, because of the first group of like letters, 60 remaining; this again, divided by two by virtue of the second pair of like letters, leaves 30 remaining. He then considers a group of six letters composed of three pairs of like letters which by the general rule would give 720 permutations. He divides that number by two, and then again by two because of the first two pairs of like letters, thus

[1]That is, $n = \frac{P_5}{P_4} = \frac{1\cdot2\cdot3\cdot4\cdot5}{1\cdot2\cdot3\cdot4} = 5.$

[2]That is, $n = \frac{P_5}{P_5} = 1.$ It is of interest to note that in the first section Cordovero considers one element as having *no* permutations.

[3]That is, $n = \frac{P_4}{P_2\cdot P_2} = \frac{1\cdot2\cdot3\cdot4}{1\cdot2\cdot1\cdot2} = 6.$

obtaining 180. For the third pair of letters, however, he divides 180 by three, obtaining.[1] He further states that if the letters of a word of eight letters, composed of four pairs of like letters, are permuted, we would divide by two twice for the first two pairs, by three for the third, and by four for the fourth. In a similar way would he find the permutations of groups containing more than four pairs of like letters.

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[1] Here Cordovero, or possibly some later scribe, is wrong. He is probably misled by the fact that if three single letters are alike, we divide by two, and then by three for the third letter. I cannot say whether or not this mistake appeared in the first edition published (see note 2). The editions of 1592 and 1786, both give this version.

THE CULTURAL VALUE OF SECONDARY MATHEMATICS

By DEAN JOHN H. MINNICK
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Before we can discuss the cultural value of secondary mathematics we must come to some conclusion as to what we mean by culture. If we turn to the works of some of the best educators of our country, we find a maze of opinions and definitions. "A more or less popular notion of the man of culture pictures him as one living apart from those who think through present-day problems and devote themselves to their solution."[1] One tells us that "Cultural education includes all forms of education; that is, training and instruction designed to develop cultural interests in such fields as art, literature, science, and history."[2] Another makes culture and discipline almost synonymous,[3] and still another defines culture as "the vital union of information and discipline."[4] Horne says that culture is "capacity for the intellectual and aesthetic enjoyment of leisure"[5] and Colvin says that "the highest culture comes to a person who has learned something useful and who has the skill to apply what he has learned."[6] Strayer and Norsworthy tell us that "It is conceivable that the person of culture is one who, by virtue of his education, has come to understand and appreciate the many aspects of social environment in which he lives; that he is a man of intelligence, essentially reasonable, and that he is willing and able to devote himself to the common good."[1] But the confusion is so great that the last named authors decided that it is best to discard the cultural aim of education. However, this aim refuses to be discarded and we find ourselves compelled to consider it.

To my mind the last statement of the meaning of culture has much to commend it. It ties up culture very definitely with usefulness. Culture, however, should mean something quite

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[1] Strayer and Norsworthy: How to Teach, Page 2.
[2] W. H. Dooley: Industrial Education, Page 2.
[3] D. E. Smith: The Teaching of Elementary Mathematics, Page 20.
[4] John Dewey: Ethical Principles Underlying Education, Page 19.
[5] Horne: Psychological Principles of Education, Page 34.
[6] S. S. Colvin: Introduction to High School Teaching, Page 339.

different from that specific usefulness which fits one for a particular position by which he makes his living and perhaps makes a very real contribution to society. It should mean a general usefulness rather than this specific usefulness. Two men may know a certain field of knowledge equally well, but one knows this field in relation to a special vocation while the second, in addition, knows it in many relations so that his outlook on life is broadened and he can see and understand the problems of other men and can enter into the life of the society of which he is a part in a fuller way. For the first man this field of knowledge has had no culture; for the second it has.

With this understanding of the subject let us turn to the consideration of some of the factors which contribute to general usefulness or culture. Among these we may consider first, a broad general knowledge of human affairs such as may be gained by travel, contact with people, and the study of history, literature and other fields of knowledge closely related to human life. That such knowledge is generally useful will scarcely be questioned. It helps man to meet others more easily and to understand their view-points better; and it helps him to interpret and master new situations as they arise. Second, an understanding of the material universe in which we live gives us a similar advantage. Man always has been and always will be interested in the natural phenomena about him and an understanding of them gives him a satisfaction and contentment which makes him more useful to society. Third, a general knowledge of how society has mastered the material world and turned it to its own use is essential to culture. If a man cannot discuss with some degree of intelligence the construction of the great bridge which is being built across the Delaware River, or at least listen with intelligence to such a discussion we can scarcely consider him cultured from our view-point. A fourth important factor is the development of one's powers and abilities generally. A man may have a particular ability developed for a specific purpose but he becomes generally useful only when all of his powers are developed in such a way that he can make various applications of them.

With this partial analysis we turn to the question, "What contribution, if any, can mathematics make to one's cultural development?"

In the first place mathematics is a great body of truth which has played an important part in the development of civilization. From the crude attempts of the Egyptian to survey his land and to construct his pyramids to the completion of the great modern engineering undertakings of today; from the crude attempts of the ancients to explain the mysteries of nature to the modern discoveries of science, mathematics has been an important factor in the advancement of society. And just as a knowledge of history and literature is essential to culture because they are important parts of human knowledge and because they give us a better insight into social conditions, just so mathematics is also necessary to culture. From this viewpoint mathematics is essential not because it is a great piece of logically organized truth, but because it is a piece of truth upon which the development of civilization has at all times been dependent. The mere study of mathematics does not necessarily carry with it the fulfilment of this phase of the cultural value of the subject. For this purpose a definite understanding of the relation of mathematics to the development of society is necessary.

Thus far we have spoken from a more or less historical viewpoint. We have referred to the relation of mathematics to the past development of society. But present day society is also dependent upon mathematics. As we will know the universe in which we live is subject to mathematical laws. Man's curiosity is still driving him on to the discovery of the hidden facts of nature. The satisfying of this curiosity is just as essential to the mental welfare of society as the satisfying of hunger is to its physical welfare. It is impossible to solve the problems of nature without an extensive knowledge of mathematics,—and a much more extensive knowledge than we can hope to give the high school pupil. But a limited amount of mathematical material properly selected and taught will give an impression of the accuracy and force with which mathematics works so

that the pupil will more readily accept the conclusion of the specialist. Here again the cultural value of mathematics depends upon the method of instruction.

In like manner, the physical comfort and the social relations of man are dependant upon methematics. Without this subject it is impossible for man to lay hold of his natural environment and adapt it to his use. Dr. David Eugene Smith[1] tells us, that if by some chance every trace of mathematical material should be removed from the world, "every mill in the whole world would slow down and every large concern would close until it could replace its accounts, its statistical material, its formulas for work, its measure, its tables, and its computing machinery, . . . every ship in the seven seas would be stricken with blindness and would wallow helpless, awaiting the probable starvation of its human burden. Not a rivet would be driven in a skyscraper in New York City. . . . Wall Street would close its portals; the engineering world would awaken tomorrow morning to a living death; the mines would shut down, and trade would relapse to the conditions of barter as in the days of savagery." No one who has taken time to reflect can doubt the accuracy of this startling picture of the world without mathematics, unless he himself is without a knowledge of the subject and its usefulness. If then the very existence of our social fabric is dependent to such a great extent upon mathematics, can one be termed cultured if he has no knowledge of mathematics and its uses to society? Yet, I find even teachers of our subject who could give a dozen instances of its relationship to present day society only with great difficulty, if at all. For them mathematics has failed to have one of the most vital phases of the cultural value and we can scarcely conceive of their pupils receiving from them that which they themselves do not have.

Some of the educators have defined the socializing value of a subject as the value derived from a subject if it gives us the right attitude toward the work of other people or of social agencies. Thus if the study of hygienic and health conditions brings an individual to co-operate with the health officers even though it means hardships through quarantine for himself, he

[1] Teachers College Record, May, 1917.

has derived a socializing value from the study. Certainly, from our view-point this is also a cultural value; for the man with the right attitude towards various social agencies is generally useful to a far greater extent than the one without such an attitude. Now it is certainly true that much of the results of the labor of the specialist even in pure mathematics has been turned to practical use and has brought good to society generally. However, frequently these specialists have not been properly encouraged because the masses of people have not had the right attitude towards them. If an able boy in your school should announce his intention of specializing in mathematics many of his teachers would discourage him, but if he should announce his desire to become a physician his decision would meet with general approval. Why this difference? Largely because the masses either do not know mathematics or they do not know its value to society. The value of a physican's work is known to all, but the works of the mathematician are hidden in electric currents, the steam engine, great bridges and other structures, all of which we accept without thinking of the men who made them possible. An important function of our instruction is to bring the individual to assume the right attitude towards the specialist upon whose work our social life depends.

For at least a small group mathematics is enjoyable. A friend tells me that often when he has a leisure hour he finds himself with paper and pencil working on some problem just for the pleasure which he derives from it. Every good teacher has found among her pupils, those who saw a beauty in mathematical forms and relations. We must remember that all pupils are individuals and as such are different. Each has his right to his own peculiar form of enjoyment. If a child should enjoy music we would never question his right to give his leisure hours to it and we would consider him all the more cultured. No one has the right to deny him the privilege of developing his musical ability. In like manner, we contend that the boy who enjoys mathematics for the sake of mathematics has an equal right to develop along his peculiar line.

Finally the man who has his mental power fully developed is generally more useful than the man who does not. The ques-

tion of discipline has been discussed so frequently that we can scarcely approach it with seriousness. However, Dr. E. L. Thorndike has given what seems to me to be a very sane statement of facts concerning this question. He says that we should not justify the place of any subject in the curriculum on the basis of formal discipline but having justified the subject on some other basis we should get all the discipline from it that is possible. If we believe that the special phase of the disciplinary value derived from the study of mathematics is to the development of the reasoning power, then I believe we will get more of the disciplinary value if we teach mathematics in its application to real life affairs in so far as high school pupils are concerned. One cannot reason in a field in which he has no experience. As much of our mathematics is now taught, it is an abstract field foreign to the child's experience. The result too often is a training in a peculiar form of memory work rather than in reasoning. Real life situations with which children are familiar will give plently of opportunity for the development of mathematics. The reasoning power will be better developed because it has been exercised and the mathematics will be no less valuable.

THE TEACHING OF ALGEBRA

By HAROLD F. RICHARDS
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I feel that teachers of algebra and authors of textbooks may well consider whether their own methods are not partly responsible for the condition which is now resulting in a wide-spread movement to erase algebra from the required curricula of secondary schools. It is a splendid gesture to lament the modern fondness of excising the backbone from education—an operation performed by leaving optional those subjects which students find difficult to master—yet a more energetic policy may be recommended when many communities are permitting pupils to forego the opportunity of satisfying the requirements for a multitude of higher courses. Algebra affords a training which tends, perhaps more than any other high school study, to develop powers of logical thinking, and it should by all means be retained everywhere as a required subject. Hearty condemnation of state superintendents is not sufficient to effect this retention; teachers will be obliged to present algebra in such manner that students can grasp it and pass it.

I have observed that by far the majority of students who fail to obtain passing grades in algebra are, nevertheless, fully able to master the fundamental operations that involve merely a mechanical application of formal rules, but fail because they cannot perform the reasoning requisite for the solution of statement problems. There is no scarcity of admirable texts which furnish full explanation of the method of solving formal problems, such as determining the value of the unknown quantity in a given equation, but I have not been able to find a text that presents a general method of attacking those problems in which the student must obtain for himself the equation to be solved. Textbooks discuss the mechanical operations necessary for the solution of the equation to which a statement problem may lead, and then append, without preliminary explanation, a series of statement problems for the student to solve. A particular solution of the first problem is usually presented in the space of one-quarter of a printed page, but the student is not given a *general method of approach.* Occasionally a suggestion is in-

cluded in the statement of an especially difficult problem, but this suggestion applies only to a limited number of similar problems, and no general mode of attack has been presented. As a result, the student reads the statement of the problem over and over, without putting anything down on his paper, and endeavors to see through all of its ramifications *before using an algebraic method to solve it.* Such an approach amounts to nothing more nor less than solving the problem without taking advantage of the tremendous power of simple algebraic analysis, and success is attained by those students who possess sufficient native shrewdness to solve the problem without algebra, while the other students fail. Every instructor will recall the testimony of pupils who say that they can work a problem by arithmetic but not by algebra, and many teachers are satisfied to accept these solutions, which are obtained rather by insight than by a rigorous algebraic method. This attitude on the part of an instructor neither extends the student's powers of analysis nor gives him a method by which he can attack problems that are not solvable by natural ingenuity.

Observation of the practice of many instructors reveals that a common method of teaching students to solve statement problems is as follows: Numbers 2, 4, 6, and 11 are assigned. The pupils arrive at class the next morning unanimous in announcing that Number 6 was too difficult for them. The instructor reads the statement of Number 6 and begins to work it on the board. Frequently he is so occupied with the business of arriving at the correct solution that he is unable to enunciate a general method which the students might have used; he obtains the answer first and then explains the problem. The pupils unite in agreeing that they understand *that* problem thoroughly; but the next morning, much to their teacher's annoyance, many of them are unable to solve a very similar problem. Obviously the instructor cannot teach students how to solve statement problems unless he himself possesses a general method by which he can read once the statement of a new example and then pass rigorously through the steps of a method which he *knows* will lead him to the correct solution.

My purpose in writing this article, however, is not so much to direct attention to this deplorable condition as to provide a

remedy. This remedy is a general method of solving statement problems. The experience which I have gained while tutoring boys from various high schools and private preparatory academies, who attend summer camps to "make up" failures in algebra, has demonstrated that the general method described below enables the pupil to attack a problem immediately after reading its statement. No time is spent by the student in pondering vaguely over the problem as a whole; he sets quickly to work and finds the answer. The method involves, as its primary principle, merely the truth that it is useful to think about one thing at a time, to form a definite conclusion expressed by a sentence containing a subject, a predicate, and a period, and then to pass on to the next fact.

The objection may be raised that it is unwise to render the solution of statement problems too simple or too mechanical, since their purpose is to exercise the student's originality. It must be remembered, however, that, if a student does not possess sufficient natural shrewdness, he will never acquire any by observing his more clever mates or his teachers use their own. The method outlined below is designed to inculcate the power of passing logically from one definite fact to another, so that the student will not befog his mind by thinking vaguely about many things at the same time without arriving at any definite conclusions whatsoever. The trouble with the average student is, that he rarely knows when he has completed the expression of a quantity or a fact; that he fails to add the period and so close his thoughts on one phase of the problem in order to pass on to the next. He endeavors to consider all at once the various facts included in the statement of a problem, and the condition of mind resulting from this effort may be likened to the confused picture obtained by superimposing several snapshots upon the same photographic film. The method described in the succeeding paragraphs will enable the student to begin the solution of a statement problem even if he does not clearly understand all of its intricacies at first, and then to continue by definite steps, each of which has a beginning and an end, rather than by flights which leave him unbalanced in mid-air. If the pupil can be induced to proceed in the solution by these definite steps, he not only will learn how to solve statement problems but will also ac-

quire a certain clarity and definiteness of judgment that will be of material benefit whether he becomes a salesman, a lawyer, a clerk, a teacher, a mechanical engineer, a housekeeper, or a writer of fiction.

I recommend that instructors devote several periods to a thorough enunciation of the following method as soon as the pupils have learned how to solve simple equations in one variable, since later difficulties may be largely avoided if the students acquire an early mastery of this general method of solving statement problems. Emphasis should here be laid upon the fact that the student makes a greater advance in working one problem by a method applicable to many others than in solving several examples by special artifices of his ingenuity.

A General Method of Solving Statement Problems

(Illustrated by instructions to the student for the solution of a typical problem involving one unknown quantity.)

The distance from Cincinnati to Hamilton is 15 miles less than the distance from Cincinnati to Oxford; and from Cincinnati to Chillicothe, which is 100 miles, is 35 miles more than the sum of the distances to Oxford and Hamilton. Find the distance from Cincinnati to Oxford.

After reading the statement of the problem, the student should *immediately* proceed to get all the facts written down in the shorthand of algebra, so that they may be manipulated and their relations seen. If a number of sticks be strewn at random in a pile, one cannot tell easily which is the shortest and which the longest; but, if the sticks are laid in an orderly row, side by side, their relative dimensions can be seen at a glance. So in algebra. The student cannot hope to solve this problem simply by reading the statement of it one or a dozen times; the solution comes *after* the problem has been re-written in algebraic notation. When all of the facts are thus arranged in regular order, their relations can easily be seen, and the student can simply glance over his tabulation and pick out two different quantities which are equal.

The Solution

Step 1. Re-write the problem in full on your paper, exactly as stated in the textbook. The reason for doing this is that you

may wish to mark up the statement and do not care to deface the book. Here it is, simply copied:

[The distance from Cincinnati to Hamilton is 15 miles less than the distance from Cincinnati to Oxford;] and from Cincinnati to Chillicothe, [which is 100 miles,] is 35 miles more than the sum of the distances to Oxford and Hamilton. [Find the distance from Cincinnati to Oxford.]

Step 2. Decide just what is to be found; and, in giving this quantity a brief name, such as X, do not forget that X always stands for a *number*, and not for cost, for amount, for silk, or for distance; X may stand for *number* of dollars, for *number* of pounds, for *number* of yards, or for *number* of miles. Here the quantity to be found is definitely stated: it is the distance from Cincinnati to Oxford. Therefore:

Let $X =$ No. of miles from Cincinnati to Oxford.

All of the words in the last sentence of the statement have now been used; mark them off with brackets or by ruling a line through them.

Step 3. Now read through the statement of the problem, noting down in algebraic shorthand every single complete fact as rapidly as you come to it. Break up the original statement into as many definite facts as possible. Starting at the beginning, we find mention of the distance from Cincinnati to Hamilton; the first idea that comes to mind is to obtain a brief expression to stand for this distance. We read the next few words and find this distance given in terms of the distance from Cincinnati to Oxford, which we have already called X. The distance in question is stated to be 15 miles less than X; and "less" always means "minus" and never anything else. Therefore we write:

Then $X - 15 =$ No. of miles from Cin. to Ham.

If we now mark off the words just used, we have left: "From Cincinnati to Chillicothe, which is 100 miles, is 35 miles more than the sum of the distances to Oxford and Hamilton." Several facts are here indicated; let us express them as quickly as possible. The first definite and complete fact is that the distance from Cincinnati to Chillicothe is 100 miles.

$100 =$ No. miles from Cin. to Chill.

NOTE: The brackets have been inserted later, in the course of the solution.

Cross out the words, "which is 100 miles," and on to the succeeding fact. The next quantity that can be set down immediately is the sum of the distances to Oxford and Hamilton. Simply glance at the tabulation and add together the expressions for these distances; if X is the number of miles from Cincinnati to Oxford, and $(X - 15)$ is the number of miles from Cincinnati to Hamilton, then

$X + (X - 15) =$ No. miles in sum of dist. to Ox. and Ham.

The next quantity mentioned, capable of immediate expression, is "35 miles more than the sum of the distances to Oxford and Hamilton." We have already expressed the sum of those distances; and "more" always means "plus" and never anything else. Therefore:

$X + (X - 15) + 35 =$ 35 more than no. miles in sum dist. to Ox. and Ham.

Re-reading the few words not marked out so far, we see that the quantity just written is also the distance from Cincinnati to Chillicothe. Indicate this fact.

$X + (X - 15) + 35 =$ No. miles from Cin. to Chill.

Every word has now been used and marked off. Therefore there is no longer any need of reading the problem as originally stated; *the solution will fall immediately from the facts as tabulated above.* These facts are collected below as they now appear on the student's paper; in this text they are separated by the explanatory words which have been inserted to show the simple reasoning that the student is performing.

$X =$ No. miles Cin. to Ox.
$X - 15 =$ No. miles Cin. to Ham.
$100 =$ No. miles Cin. to Chill.
$X + (X - 15) =$ No. miles in sum of dist. to Ox. & Ham.
$X + (X - 15) + 35 =$ 35 more than no. miles in sum dist. to Ox. & Ham.
$X + (X - 15) + 35 =$ No. miles Cin. to Chill.

Now glance over the right-hand members of the statements tabulated above, and find two lines which are identical. It is observed at once that the number of miles from Cincinnati to Chillicothe is 100 and is also $X + (X - 15) + 35$; therefore these two quantities must be equal to each other, since things equal to the same thing are equal to each other. We thus have a simple equation at once:

$$X + (X - 15) + 35 = 100$$

The problem is nearly solved now, for all of the reasoning has been performed and nothing remains but the mechanical working out of this equation by applying rules that can easily be remembered by any student who can remember that twice four is eight. Applying these rules, we find that

$$2X = 80$$

If 2 cows cost \$80, one cow costs \$80 divided by 2, which is \$40; if $2X$ equal 80, one X equals 80 divided by 2, which is 40. Hence:

$$X = 40$$

Thus the distance from Cincinnati to Oxford is 40 miles. In order to prove that this answer is correct, we return to the original statement of the problem. The distance to Hamilton is 15 miles less than 40 miles, or 25 miles, and the sum of these two distances is 65 miles. The distance to Chillicothe, or 100 miles, *is* 35 more than 65; hence the answer satisfies the problem as stated and we conclude that no mistake has been made.

* * *

The teacher will find no difficulty in applying this method to the solution of statement problems involving two or three variables or leading to quadratic equations. The foregoing explanation shows that the method is perfectly general and applicable to any of the statement problems found in standard texts. The solution of a determinate problem falls out at once when *all* of the facts are briefly tabulated, for the student will always find two different expressions for one of the quantities. Furthermore, this method immediately detects any "trick" problems; I have found teachers who have been unable to show that a problem of such nature was indeterminate. The student has here a definite, general method of attack, which enables him to avoid the mental confusion and waste of time that usually result from fruitlessly pondering over the problem as a whole; he is able to set immediately to work and advance steadily towards the correct solution. He no longer looks upon algebra as a collection of peculiar puzzles, but is induced by the inevitable success attained with this method to regard mathematics as a useful tool.

CONCERNING THE INTERCOMMUNION OF MATHEMATICS AND ASTRONOMY

By AGNES G. ROWLANDS

Plato in his Republic said, "Now when all these studies (Arithmetic, Geometry and Astronomy) reach the point of intercommunion and connection with one another, and come to be considered in their mutual affinities, then I think, but not till then, will the pursuit of them have a value for our objects."

Secondary school mathematics teachers have perhaps not always realized the powerful ally they have in astronomy in helping their students to appreciate the underlying meaning and significance of their mathematics, without which the students cannot possibly get all the cultural and spiritual values from their work. With great joy, I have watched the reaction, again and again, of mathematics classes when I have told them, for example, about the discovery of the planet Neptune through the mere computation of two mathematicians. In 1781 Sir William Herschel discovered the planet Uranus, the first one to be discovered with the telescope. The planet was observed for a sufficient length of time to enable astronomers to plot its theoretical path. In presenting this fact, students derive real pleasure in realizing that their work in graphs and plotting equations enables them to understand how observations of the planet's changing positions through a limited part of her period of revolution around the sun gives enough "values" to make possible the plotting of the planet's orbit. But Uranus strayed from her theoretical path thus mathematically mapped out, by 2″ of arc. One might think that such a slight discrepancy between theory and practice as 2″ of arc, was not worth minding. But this "intolerable" discrepancy, and the Mathematical necessity for explaining it, led to the discovery of a new planet. Evidently Uranus departed from her path because some undiscovered planet was "pulling" her from her path. To find this planet it was useless to search the sky at random with a telescope, because of the great number of fixed stars. The position of this undiscovered planet must first be determined. This could only be done by mathematics. Two mathematicians,

Adams in England, and Leverrier in France, set to work to solve the problem. They knew the position of Uranus. They knew how much she was pulled out of her theoretical path, and they knew the law of attraction. Here even ninth grade pupils working with formulas, early have felt a great satisfaction in realizing that their little work in formulas enabled them to follow a discussion using the formula $G = \frac{MM^1}{d^2}$, where G stood for attraction, M and M^1 the masses of two mutually attracting bodies, and d the distance between the attracting bodies. Mathematicians calculated that if a planet of a certain size, traveling in a certain orbit, at a certain speed, existed, it would explain the behavior of Uranus. They also calculated that if this planet existed, it would be visible at a certain time in a certain part of the sky. Of course, the actual calculation was exceedingly difficult. Adams completed his calculations even earlier than Leverrier, but the English did not have, as the Germans did, a recently made accurate map of the stars in the suspected part of the sky. Leverrier sent his calculations to a German astronomer named Galle, writing in substance:

"Direct your telescope to a point on the ecliptic in the constellation of Aquarius, in longitude 326°, and you will find within a degree of that place a new planet, looking like a star of the ninth magnitude, and having a perceptible disc."

The planet was found in Berlin September 23, 1846, in exact accordance with this prediction, within half an hour after the astronomers began looking for it, and within 52′ of the precise point that Leverrier had indicated. Ponder the significance of the discovery—a mathematician in his study backed by his knowledge of and faith in the mathematical laws in obedience to which he had reason to believe heavenly bodies moved, able to direct astronomers where to find a hitherto undiscovered body 2,800,000,000 miles from the earth. And what is more, later observations showed the mathematician's calculation of the planet's mass to be correct. Of the reasons for the discrepancy between the calculations of the planet's distance and its observed distance, we shall speak presently in connection with Bode's law.

Bode's law gives pupils in mathematics a sense of appreciation of the "Mathematical Character" of the universe, hardly less striking than the Mathematician's discovery of the planet Neptune. Write a series of 4's, under the names of the planets.

Venus	Mercury	Earth	Mars		Jupiter	Saturn	Uranus	Neptune
4	4	4	4	4	4	4	4	4
	3	6	12	24	48	96	192	384
.4	.7	1.00	1.6	2.8	4.8	10.0	19.6	38.8

Under the second 4 write 3, the next one 6, 12, and so on in geometric progression. Divide the sum by ten. The quotient gives the approximate distance of the planets from the sun in astronomical units (an astronomical until being 93,000,000 miles). Thus, according to this law the earth is one astronomical unit from the sun, Mercury is .7 of an astronomical unit or 6,510,000 miles. Observation gives approximately this distance for Mercury. But it was found that Jupiter was at a distance not of 2.8 astronomical units, but of 4.8 astronomical units, Saturn of 10.0 astronomical units. What did astronomers and mathematicians conclude—why, that at the distance of 2.8 astronomical units, there must be a hitherto unknown planet. An association was formed to search for the missing planet. Four very small planets were found at the approximate distance called for by Bode's law. Since then many other small planets or asteroids have been discovered in this region of the sky—more than nine hundred now being known. The fact that Bode's law breaks down with Neptune, as well as some other facts in regard to the planet, has led astronomers and mathematicians to believe there must be another planet out beyond Neptune, and they are working in the hope of discovering this planet.

I have already referred to the satisfaction derived by the students from the study of graphs in the light which this study throws on gaining knowledge in regard to Uranus. The study of graphs is perhaps one of the easiest ones we have to make vital and real to the student, except when we come to the graphing of equations, especially equations of degree higher than the first. Here astronomy should be utilized. Compare graphs of type forms of equations with the orbits of heavenly bodies.

Show orbits of comets formerly thought to travel in hyperbolic or parabolic parts. These curves will have a new meaning when the students realize that an understanding of them enables them to appreciate why some comets have never been seen but once, and others, like Halley's comet, return at regular intervals. Have the pupils find out about the recent discoveries of astronomers and mathematicians showing that all comets really travel in elliptical orbits, and so are members of our solar system, but in many cases the ellipses are so elongated, that paths of these comets were for a long time thought to be parabolas or hyperbolas. And that was why comets were erroneously thought to come from out beyond our solar system.

There are of course, many more illustrations which I might give, such as the determination of the velocity of light thru mathematical calculations in connection with the eclipsing of Jupiter's satellites. But enough has been said to illustrate the way in which astronomy ought to be regularly the ally of mathematics, in getting a true comprehension of the full significance of our subject. Do you think a class would forget that an exterior angle of a triangle was equal to the sum of the two remote interior angles, if shown how knowledge of this relationship made it possible for Erastosthenes (27 5-194 B. C.) to compute the circumference of the earth with a fairly close approximation to the true value? But after all, the real purpose of showing the "intercommunion" of the mathematics and astronomy is not merely a practical one of "driving home" the Mathematics itself. Its real purpose is helping the pupils to get the cultural and spiritual values from the mathematics work. This is only possible with an appreciation of the significance of mathematics, and a realization of its "world relationships," so that the student sees that a mathematical formula may be a way of expressing an "eternal verity" at the heart of the universe; or, as Maria Mitchell, one of the greatest of women astronomers put it, "a Mathematical formula is the hymn of the universe."

THE ART OF QUESTIONING[1]

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In the study of the history of educational systems from the earliest times to the present, it is evident that the question has always been a factor in teaching. Teaching in Grecian and Roman times produced, by means of the question, thinkers and philosophers. In the middle ages, the question played an important part in the teaching which was mostly to the individual and not to groups.

In our own country, the secondary schools are in their third stage of development. We first had the grammar schools, then the academies, and then the publ' high school. In each of these the teaching has been done to groups and not to individuals. At present the school population in most towns has so outgrown the school facilities that the groups reporting to teachers are much too large. So it remains for the teacher to use such devices or means as he can, for the betterment of the groups before him.

One of the factors is questioning. Questioning is of two kinds: (1) to test whether pupils have prepared or mastered certain material, and (2) to stimulate and guide their thinking along new lines. The teacher needs as much skill in one kind of questioning as in the other. To do either skillfully he must understand the types of learning. Parker gives for practical purposes the following types of learning.

1. Acquiring motor skill, as in gymnastic and shop activities, in dancing, in musical technique, and in the pronunciation of a foreign language.

2. Associating symbols and meanings, as in learning vocabularies, the teacher using objects, charts, etc.

3. Acquiring habits of enjoyment, as training in the enjoyment of sports and games, music, literature, drama, and art.

4. Acquiring skill in expression, both oral and written.

5. Acquiring skill in reflective thinking.

Reflective thinking is one of the most important things in mathematics. In the form of problem-solving, and the working of examples, reflective thinking plays a large part.

[1] Delivered at the mid-winter of the Association of Teachers of Mathematics in New England.

The teacher may direct and assist the pupils in their thinking by teaching them to define their problems and keep them in mind. Some pupils will fail to see any problem at all. They "sit tight" mentally and are not disturbed.

Others realize that something must be done but do not know what. They assume a worried look, wring their hands, and then perhaps do something impulsive.

A third type sees the question at once, and immediately proceeds to get to the bottom of it.

When the problem is once defined the pupil must then be taught to keep it in mind.

One of the chief attributes of a skillful presiding officer of an assembly is his ability to keep the discussion to the problem before the house. Again, let me repeat, the teacher may help the pupils in their thinking by teaching them to define their problems and keep them in mind.

The teacher may also direct and assist the pupils in their thinking by teaching them to formulate hypotheses and to recall principles. Thorndike says, "the wise course is not to eliminate altogether the independent search by pupils for the proper class, but to make it easier and briefer by directing it. It is made easier by systematizing the process of search, by limiting the number of classes among which the pupil must search, by informing him of classes which include the right one and which he would neglect if undirected, and by calling his attention to the consequences of membership in a certain class."

The teacher may also direct and assist the pupils by teaching them to organize their thinking. The pupil can be taught to take stock of their inquiry from time to time and to keep their results carefully tabulated. If the thinking is done by a class, it is well to develope an outline on the blackboard.

In doing these things the teacher at the same time gives the pupils opportunities to reason independently, and to participate in group reasoning, and to follow and supplement the teacher's reasoning.

With these high ideals in mind, the teacher, conducting a recitation, finds himself in a complicated situation. He never knows just what move to expect. He must be ever alert to devise appropriate questions and to adapt himself to circumstances.

Among the chief elements of successful questioning are clear and rapid thinking. To think clearly one must be thoroughly acquainted with his subject. He must be able to make clear-cut distinctions, comparisons, and classifications. Rapidity is as necessary as clearness. Slow, clear thinkers would probably prepare excellent written questions, but might fail utterly in oral questioning.

Successful questioning also involves a sense of relative values. A teacher should be able to determine quickly just what use to make of each response to a question, whether it should be followed up or passed over. Here again a good knowledge of the subject is a necessity.

A third important factor is skill in wording questions. Skill in expression is as important as the clear and rapid thinking or the sense of relative values. A good lecturer might be a poor questioner. The questions should be so clear and simple that there will be no doubt in the pupils' mind as to the meaning. Otherwise there may not be the desired concentration of thought upon the answer. As a rule, a question should be as concise and definite as the answer it solicits. The question should be the outgrowth of the teacher's knowledge. The teacher should be so full of his subject that the flow of questions will be rapid and logical.

Every question should have a definite purpose. In mathematics it should not be necessary for the teacher to depend upon prepared questions. It would be well to plan a general scheme for the day's recitation, but not to actually prepare the questions as might be done in teaching History, or some other subjects.

In Geometry, for example, a series of questions would lead to a definite end. Questions could be asked concerning the analysis of the theorem, the construction of the figure, the reason for the equality of triangles, the purpose of proving the equality of triangles, and the conclusion. These questions could well be followed by questions on similar truths, such as are usually found in the corollaries or exercises immediately following a theorem.

In Algebra, the questions are usually independent of each other. They are short, and lend themselves to rapid drill. For example, in the work on exponents one can easily have rapid drill in squaring binomials in which the terms have negative, fractional, or literal exponents. Questions in factoring, multiplication, division, raising to powers, extracting roots, symplifying surds, and such subjects make good material for oral drill.

As a general rule it is well to avoid all "yes" and "no" questions, or questions which do not stimulate thought. Any question which suggests the answer is worthless. Whatever the pupil's ignorance, he is more than likely to answer these worthless questions correctly. The teacher's manner of asking the question, and the hints of fellow pupils make guessing very easy. The pupil is deceived in that he thinks he knows something when he does not. He is also probably fostering bad habits of study.

In order to avoid inattention, it is well not to repeat questions. If the question is short, pithy, and well worded there will be no need for repetition. It sometimes happens that after a question has been asked that even the best pupils may look puzzled. If so, the question was probably poorly worded. The teacher should then reward his question without having asked for a response.

Practically all of our questioning is to the group. So our questions should be addressed to the group and not to the individual. In no case should a pupil be called upon to stand and then the question asked. Under such conditions the rest of the class might become dreamers. The correct method would be to first ask the question, then give a brief time for thinking, and finally call upon some pupil for the response.

While the pupils of the group may be arranged alphabetically in their seats, or in some other regular order, they should not be called upon in order. Try to find some devise for securing a fair distribution of questions. Some teachers put the pupils names on cards and shuffle these before each recitation. Almost any method will do as long as there is a method.

The lack of method would be a real detriment to a class and to the individual. In an investigation in one of the western high schools, it was found that in a Mathematics class of 21

pupils, one student was called on twice and another eleven times in nine days. In another class of thirteen pupils, one pupil was questioned four times and another eighteen times. In a Geometry class the range was from six to twenty three recitations in ten days. It is probable that some irregularities cannot be avoided, but as far as passible we should try and not have such great differences.

Time should not be wasted in assisting the poor pupil too much or in pursuing the talkative brilliant pupil. If a pupil is very weak, he should be helped during a period of supervised study or during the teacher's office hours, but in no case should the time for drill be wasted.

While I have had no experience with "flash-cards," I am told that they are a great help in rapid oral drill on many of the processes in Algebra.

In closing, let me quote Prof. Harris, "So in the matter of questioning, one could accomplish much for instruction in a very short time if he did nothing more than to insist upon introducing into the plan of every lesson a short series of related questions calling for reflection. The results that might reasonably be expected are these, Where six or eight purposeful questions are asked and satisfactorily answered, the number of questions will be reduced, the pace will become more normal, pupils will be forced to tie up their facts in profitable relations, the several questions will serve as high lights in the lesson, pupils will have practice in the habit of studying a lesson for the salient points, and they will eventually grow into the habit of organizing subject matter for themselves. With such attainments as these we should have some positive factors to deal with in measuring the efficiency of instruction."

NEW BOOKS

Farm Projects. By Carl Colvin and John A. Stevenson. Macmillan and Company, 1922, New York. Pp. 363.

An Introduction to the Graphic Language. By Gardner C. Anthony. D. C. Heath and Company, New York and Chicago. Pp. 158.

Mathematics for Technical and Vocational Schools. By Samuel Slade and Louis Margolis. Edited by Joseph M. Jameson. John Wiley and Sons. New York. Pp. 491.

Industrial Physics: Mechanics. By L. Raymond Smith. McGraw Hill Book Company, New York. Pp. 226.

General Mathematics, Book Two; by William David Reeve. Ginn & Co. $1.60.

This book is intended to follow Book One of the same title by Schorling and Reeve. The preface says that the work has been thoroughly and successfully tested as work for the tenth grade.

The geometry included covers the plane geometry requirement for college, with profuse and interesting exercises and applications, well classified. Serious effort is made to teach methods of attack on a new theorem, and to guard against the common lapses of logic. Some "optical illusions and geometric fallacies" are given to help in this general direction. There is a small fragment of solid geometry, including the traditional proof for the perpendicular to two straight lines at their intersection, and some exercises of construction not in one plane.

The sections devoted to trigonometry include the solution of right and oblique triangles, a three-place table of natural sines, cosines, and tangents being used for all computations. Formulas are established for radii of circumscribed and inscribed circles.

Under Algebra are given fractions and the sacred "theorems of proportion." The author seems a little timid about discarding the cumbrous algebra of the so-called "geometric proof." Elimination in linear-quadratic pairs, and in certain convenient quadratic pairs, is treated in connection with the conic sections, which are defined as loci in the usual fashion.

Arithmetic, the preface claims, is kept before the pupil by means of reviews and applications. Wise caution is given, pp.

319-320 (late in the game!), on approximate calculation; but the author systematically neglects his own maxims, obtaining results to four or five figures were his data are of two or three places.

Comment is plentiful, and very instructive, detailed, and interesting. This is especially true in connection with similarity and scale drawings. Portraits, of the usual irrelevance and stupidity, appear at discreetly separated pages.

On the whole, the book is extreely suggestive. In the hands of a watchful teacher, it would be a text-book of very great value; and for any teacher it would be a book of reference full of material for new departures in teaching. It should certainly be in the hands of every teacher of geometry.

Such books as this, that open new lines of work in teaching, are peculiarly subject to errors, and sometimes to disconcerting inaccuracies, which the humdrum, old-fashioned book will automatically avoid. These things result from inadvertence, the author's mind being preoccupied with the development of his newer ideas of teaching.

The index here is not entirely adequate. The many changes from the customary alignment of topics make a detailed index even more necessary than ever.

George W. Evans,
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NEWS NOTES

The Fourth Annual Meeting of the National Council of Teachers of Mathematics

President John H. Minnick, and the Executive Committee have arranged to hold the fourth annual meeting of the Council in Cleveland, Ohio, February 28, 1923.

The program consists of:

Business Meeting, 10:30 A. M., Lecture Room, Hotel Olmsted.

Executive Committee Meeting, 12:00 M. Luncheon, Hotel Olmsted.

Afternoon Program, Hotel Olmsted, 2:00 P. M.

Demonstrative Geometry as a Subject for Training. D. W. Werremeyer, West Technical High School.

Mathematics of the Junior High School. Miss Amy Preston, Roosevelt Junior High School, Columbus.

Results Obtained from the Roger's Test of Mathematical Ability. Dr. Agnes Rogers, Goucher College, Baltimore.

Banquet 6:30 P. M. Hotel Winton

President Robert J. Aley, Butler College, presiding.

Organization of the Mathematics of the Seventh, Eighth and Ninth Grades According to Pedagogical Units. E. R. Breslich, School of Education, University of Chicago.

Teaching the Algebraic Language to Junior High School Pupils. J. R. Overman, Ohio State Normal, Bowling Green.

The cost of the Banquet is $1.65 per cover. Reservations should be sent to Mr. D. W. Werremeyer, West Technical High School, Cleveland, Ohio.

All persons who are interested in the teaching of secondary mathematics are invited to attend the meetings. A large attendance is anticipated.

The seventh annual meeting of the Mathematical Association of America was held at Harvard University, Cambridge, Mass., on Thursday and Friday, December 28 and 29, 1922, in affiliation with the American Association for the Advancement of Science and in connection with the annual meeting of the American Mathematical Society. The program included:

WEDNESDAY AFTERNOON

1. A Symposium on Space and Time. Professor G. D. Birkhoff, P. W. Bridgman, and Harlow Shapley, of Harvard University, spoke respectively on *The Logic of Space and Time*, *The Physical Meaning of Space and Time*, *The Astronomical Measures of Space and Time*.

THURSDAY AFTERNOON

2. *Reduction of Singularities of Plane Curves by Birational Transformation.* Professor G. A. Bliss, University of Chicago, retiring president of the American Mathematical Society.

3. *The Grafting of the Theory of Limits on the Calculus of Leibniz.* Professor Florian Cajori, University of California, representing the Mathematical Association of America.

4. *Geometry and Physics.* Professor Oswald Veblen, Princeton University, retiring vice-president of Section A of the American Association for the Advancement of Science.

FRIDAY MORNING

5. Annual business meeting and election of officers.

6. *Period of the Bifilar Pendulum for Finite Amplitudes.* Professor H. S. Uhler, Yale University.

7. *Skew Squares.* Professor W. H. Echols, University of Virginia.

8. *On the Averaging of Grades.* Professor C. F. Gummer, Queen's University.

9. *Mathematics at Oxford and the Ph. D. Degree.* Professor W. R. Burwell, Brown University.

10. *Some Unsolved Problems in the Theory of Sampling.* Professor B. H. Camp, Wesleyan University.

11. *Some Unsolved Problems in Solid Geometry.* Professor J. L. Coolidge, Harvard University.

FRIDAY AFTERNOON

12. *The Subject Matter of a Course in Mathematical Statistics.* H. L. Rietz, Head of the Department of Mathematics at the State University of Iowa, and chairman of the National Research Council's Committee on the Mathematical Analysis of Statistics.

13. *Time Series of Economic Statistics: Their Fluctuation and Correlation.* Warren M. Persons, Professor of Economics at Harvard University, and Editor of the *Review of Economic Statistics* published by the Harvard Committee on Economic Research.

14. *Some Fundamental Concepts of the Calculus of Mass Variation and Their Relations to Practical Problems.* Arne Fisher, author of *The Mathematical Theory of Probabilities and its Application to Frequency Curves and Statistical Methods.*

NEWS AND NOTES

President JOHN H. MINNICK and the Executive Committee of the National Council of Teachers of Mathematics have perfected plans to present the interests of the Council to practically all organizations of mathematics teachers in the United States. State Representatives have been appointed, as follows:

Alabama—Frank Ordway, Central High School, Birmingham
Arizona—A. L. Hartman, Mesa Union High School, Mesa
Arkansas—George W. Drake, University of Arkansas, Fayetteville
California—Gertrude E. Allen, University High School, Oakland
Colorado—E. L. Brown, Northside High School, Denver
Delaware—Mrs. Elinor B. Rosa, Milford
District of Columbia—Harry English, Board of Examiners, Washington
Florida—Miss Olga Larson, Box 84, Apopka
Georgia—George W. Brindle, Surrency
Idaho—Winona M. Perry, 719 Sherman Ave., Couer D'Alene
Illinois—R. L. Modesitt, 1703 S. 7th St., Charleston
Indiana—Walter G. Gingery, Shortridge High School, Indianapolis
Iowa—Ira S. Condit, Iowa State Teachers College, Cedar Falls
Kansas—Miss Inez Morris, 728 State St., Emporia
Kentucky—V. D. Roberts, Somerset
Louisiana—Jeanne Vautrain, 1820 N. Rampan St., New Orleans
Maine—E. L. Moulton, Edward Little High School, Auburn
Maryland—Miss N. V. Orcutt, Girls' Latin High School, Baltimore
Massachusetts—William H. Brown, High School, Amherst
Michigan—John P. Everett, Western State Normal School, Kalamazoo
Minnesota—W. D. Reeve, 828 University Ave., Minneapolis
Mississippi—Miss Clyde Lindsey, Oxford
Missouri—Charles Ammerman, McKinley High School, St. Louis
Nevada—Miss Bertha C. Knemyer, Elko Co., High School, Elko
New Mexico—T. C. Rogers, 1018 Fourth St., E. Las Vegas
New York—Raleigh Schorling, 423 West 123rd St., New York City
Ohio—Miss Florence M. Brooks, Fairmount, Jr. High School, Cleveland
Oklahoma—C. E. Herring, Box 489, Oklahoma City
Oregon—Florence P. Young, Frankline H. S., Portland
Rhode Island—P. S. Crosby, 110 N. Bend St., Pawtucket
South Carolina—J. Bruce Coleman, University of South Carolina, Columbia
South Dakota—Iona J. Rehn, 735 S. Summit Ave., Sioux Falls
Tennessee—F. L. Wrenn, McCallie School, Chattanooga
Texas—J. O. Mahoney, 1900 Crockett St., Dallas
Vermont—Llewellyn R. Perkins, 6 Franklin St., Middlebury
West Virginia—Miss Blanche Stonestreet, 591 Spruce St., Morgantown
Wisconsin—Miss Mary A. Potter, Racine High School, Racine

These representatives are actively engaged in urging the teachers in their respective states to affiliate with the Council, and to participate, in a more direct way, in the reorganization movement now being effected in mathematical education. A special circular has been prepared to set forth the purposes and values of the Council. Copies may be secured from your representative, from Mr. John A. Foberg, Secretary-Treasurer, Camp Hill, Pa., or from President John H. Minnick, School of Education, University of Pennsylvania, Philadelphia.